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Recurrence relations for connection coefficients between Q-orthogonal polynomials of discrete variables in the non-uniform lattice $x(s) = q^{2s}$

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Abstract. We obtain the structure relations for q-orthogonal polynomials in the exponential lattice q^{2s} and from these we construct the recurrence relation for the connection coefficients between two families of polynomials belonging to the classical class of discrete q-orthogonal polynomials. An explicit example is also given.

1. Introduction

Given two families of polynomials, denoted by $P_n(x)$ and $Q_m(x)$, of degree exactly equal to n and m, respectively, the connection problem asks one to compute the so-called connection coefficients $C_m(n)$ defined by the relation

$$P_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x).$$

When both families are *orthogonal* with respect to two different measures the connection coefficients satisfy a relatively simple recurrence relation, but mixing in the (m, n) table three adjacent m and three adjacent n crossing at (m, n).

The first survey on this topic was given by Askey 20 years ago [1, 2], which gave in some cases explicit expressions for the coefficients and also discussed the positivity properties of these coefficients.

It was noticed only recently that an additional assumption on the orthogonality measure gives for $C_m(n)$ a recurrence only in m, n being fixed. This orthogonality class is called *semi-classical* and is very large [11, 7]. The classical (continuous) family, Jacobi, Bessel, Laguerre, Hermite (see for instance [12, 5]) and the classical (discrete) family, Hahn, Kravchuk, Meixner, Charlier (see for instance [13, 5]) are of course included in the semi-classical class. When the orthogonality measure is defined by a weight $\rho(x)$, the semi-classical class covers all weights which are solutions of a linear first-order differential (or difference) equation with polynomial coefficients.

The key property inside the semi-classical class, in order to obtain a one index (m) recurrence relation for $C_m(n)$, comes from the existence of a so-called structure relation

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linking linearly the derivative (or difference) of $P_n(x)$ times a polynomial to a fixed combination of $P_k(x)$.

An algorithm has been recently given which builds for both discrete and continuous classical families (see [3], [15] and [16]) the explicit recurrences for $C_m(n)$, solving in many cases these recurrences with the help of *Mathematica* [20] (see also [10, 18, 19]).

Searching for the situation for which a structure relation is known explicitly, we realize that, from the data of the orthogonal polynomial on the exponential lattice $x(s) = q^{2s}$ (a small subset of the q-world). Here we need to point out that there exists two different points of view in the study of q-polynomials: the first one, in the framework of the q-basic hypergeometric series [6, 8, 9] and the second, in the framework of the theory of difference equations developed by Nikiforov *et al* [12–14]. In this work we will use the second method because it gives us the possibility of providing uniform treatment of several classes of orthogonal polynomials and, probably, it is the best way to find further applications.

This paper shows how to apply the technique to a particular (simple) case: the exponential lattice, building first the corresponding structure relations.

2. Structure relations for q-orthogonal polynomials on the exponential lattice $x(s) = q^{2s}$

Let us start with the study of some general properties of orthogonal polynomials of a discrete variable in non-uniform lattices. Let

$$\tilde{\sigma}(x(s))\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla Y(s)}{\nabla x(s)} + \frac{\tilde{\tau}(x)(s)}{2}\left[\frac{\Delta Y(s)}{\Delta x(s)} + \frac{\nabla Y(s)}{\nabla x(s)}\right] + \lambda Y(s) = 0$$
$$\nabla f(s) = f(s) - f(s-1) \qquad \Delta f(s) = f(s+1) - f(s) \tag{1}$$

be the *second-order difference equation of hypergeometric type* for some lattice function $x(s)$, where $\nabla f(s)$ and $\Delta f(s)$ denote the backward and forward finite difference quotients, respectively. Here $\tilde{\sigma}(x)$ and $\tilde{\tau}(x)$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ is a constant. Equation (1) can be obtained from the classical *hypergeometric equation*

$$\tilde{\sigma}(x)y''(x) + \tilde{\tau}(x)y'(x) + \lambda y(s) = 0$$

via the discretization of the first and second derivatives y' and y'' in an appropriate lattice [12, 13]. It is better to rewrite (1) in the equivalent form (see [13, 14])

$$\tilde{\sigma}(s)\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla Y(s)}{\nabla x(s)} + \tau(s)\frac{\Delta Y(s)}{\Delta x(s)} + \lambda Y(s) = 0 \tag{2}$$
$$\sigma(s) = \tilde{\sigma}(x(s)) - \tfrac{1}{2}\tilde{\tau}(x(s))\Delta x(s-\tfrac{1}{2}) \qquad \tau(s) = \tilde{\tau}(x(s)).$$

The q-orthogonal polynomials $P_n(x(s))_q \equiv P_n(s)_q$ on the exponential lattice $x(s) = q^{2s}$ are, for given functions $\sigma(s)$ and $\tau(s)$, the polynomial (in powers of $x(s) = q^{2s}$) solutions of the second-order difference equation (2).

The k-order difference derivative of the polynomials $P_n(x(s))_q$, defined by

$$v_{kn}(s) - \frac{\Delta}{\Delta x_{k-1}(s)}\frac{\Delta}{\Delta x_{k-2}(s)}\cdots\frac{\Delta}{\Delta x(s)}[P_n(x(s))_q] \equiv \Delta^{(k)}[P_n(x(s))_q]$$

and

$$x_m(s) = x(s + m/2)$$

also satisfy the difference equation of hypergeometric type of the form

$$\sigma(s)\frac{\Delta}{\Delta x_k(s-\frac{1}{2})}\left[\frac{\nabla v_{kn}(s)}{\nabla x_k(s)}\right]+\tau_k(s)\frac{\Delta v_{kn}(s)}{\Delta x_k(s)}+\mu_k v_{kn}(s)=0 \qquad (3)$$

where (see [13], p 62, equation (3.1.29))

$$\tau_k(s)=\frac{\sigma(s+k)-\sigma(s)+\tau(s+k)\Delta x(s+k-\frac{1}{2})}{\Delta x_{k-1}(s)}$$

and

$$\mu_k=\lambda_n+\sum_{m=0}^{k-1}\frac{\Delta\tau_m(s)}{\Delta x_m(s)}.$$

These polynomial solutions denoted by $P_n(x(s))_q \equiv P_n(s)_q$ satisfy the orthogonality property

$$\sum_{s_i=a}^{b-1}P_n(x(s_i))_q P_m(x(s_i))_p\rho(s_i)\Delta x(s_i-\tfrac{1}{2})=\delta_{nm}d_n^2 \qquad (4)$$

where $\rho(x)$ is some non-negative function (weight function), i.e.

$$\rho(s_i)\Delta x(s_i-\tfrac{1}{2})>0 \qquad (a\leqslant s_i\leqslant b-1)$$

supported in a countable subset of the real line $[a,b]$ (a,b can be $\pm\infty$). The functions $\rho(s)$ and $\rho_k(s)$ are the solutions of the Pearson-type difference equations ([13], p 64, equations (3.2.9) and (3.2.10))

$$\frac{\Delta}{\Delta x(s-\frac{1}{2})}[\sigma(s)\rho(s)]=\tau(s)\rho(s) \qquad (5)$$

and

$$\frac{\Delta}{\Delta x_k(s-\frac{1}{2})}[\sigma(s)\rho_k(s)]=\tau_k(s)\rho_k(s) \qquad (6)$$

and $\rho(s)$ satisfy the condition [14]

$$\sigma(s)\rho(s)x^k(s-\tfrac{1}{2})|_{s=a,b}=0 \qquad \forall k,l\in\mathbb{N} \qquad (\mathbb{N}=0,1,2,\ldots\}).$$

In (4) d_n^2 denotes the square of the norm of the corresponding orthogonal polynomials.

The q-orthogonal polynomials satisfy a three-term recurrence relation (TTRR) of the form

$$x(s)P_n(s)_q=\alpha_n P_{n+1}(s)_q+\beta_n P_n(s)_q+\gamma_n P_{n-1}(s)_q \qquad (7)$$

with the initial conditions

$$P_{-1}(s)_q=0 \qquad P_0(s)_q=1.$$

It is well known [13, 14] that the polynomial solutions of equation (2), denoted by $P_n(x(s))_q$, are uniquely determined, up to a normalizing factor B_n, by the difference analogue of the Rodriques formula (see [13], p 66, equation (3.2.19)):

$$P_n(s)_q=\frac{B_n}{\rho(s)}\nabla_n^{(n)}[\rho_n(s)] \qquad \nabla_n^{(n)}=\frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_n(s)}[\rho_n(s)] \qquad (8)$$

where $\rho_n(s)=\rho(n+s)\Pi_{k=1}^n\sigma(s+k)$. These solutions correspond to some values of λ_n, eigenvalues of equation (2), which are computed from (see [13], p 104, [14])

$$\lambda_n=-\tfrac{1}{2}[n]_q\{(q^{n-1}+q^{-n+1})\tilde{\tau}'+[n-1]_q\tilde{\sigma}''\} \qquad (9)$$

where $\tilde{\sigma}(s) = \sigma(s) + \frac{1}{2}\tilde{\tau}(s)\Delta x(s-\frac{1}{2})$ and $\tilde{\tau}(s) = \tau(s)$ (see equation (2)).

Here $[n]_q$ denotes the so-called q-numbers

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{\sinh(hn)}{\sinh(h)} \qquad q = \mathrm{e}^h.$$

2.1. The first structure relation for the q-polynomials in the lattice $x(s) = q^{2s}$

Let us now try to obtain a structure relation for the q-polynomials in the exponential lattice $x(s) = q^{2s}$. (For the linear lattice see [13], p 24, equation (2.2.10).)

First, we rewrite the Rodriques equation (8) in another form. We will use the linearity of the operator $\nabla_n^{(n)}$, as well as the identity

$$\nabla x_k(s) = q^k \nabla x(s).$$

Then, a straightforward calculation gives us

$$P_n(s)_q = \frac{q^- n(n+1)/2 B_n}{\rho(s)} \left[\frac{\nabla}{\nabla x(s)}\right]^n [\rho_n(s)] \qquad \left[\frac{\nabla}{\nabla x(s)}\right]^n [\rho_n(s)]$$
$$= \overbrace{\frac{\nabla}{\nabla x(s)} \cdots \frac{\nabla}{\nabla x(s)}}^{n\text{-times}}. \tag{10}$$

Now, from formulae (5) and (10) we find

$$\frac{\nabla \rho_{n+1}(s)}{\nabla x_{n+1}(s)} = \frac{\nabla[\rho_n(s+1)\sigma(s+1)]}{\nabla x_n(s+\frac{1}{2})} = \frac{\Delta[\sigma(s)\rho_n(s)]}{\Delta x_n(s-\frac{1}{2})} = \tau_n(s)\rho_n(s).$$

Then by using the Rodriques formula (8) we obtain

$$P_{n_1}(s)_q = \frac{B_{n+1}}{\rho(s)} \nabla_{n+1}^{(n+1)}[\rho_n(s)] = \frac{B_{n+1}}{\rho(s)} \nabla_n^{(n)} \frac{\nabla \rho_{n+1}(s)}{\nabla x_{n+1}(s)} = \frac{B_{n+1}}{\rho(s)} \nabla_n^{(n)}[\tau_n(s)\rho_n(s)]$$
$$= q^{-n(n+1)/2} \frac{B_{n+1}}{\rho(s)} \left[\frac{\nabla}{\nabla x(s)}\right]^n [\tau_n(s)\rho_n(s)]. \tag{11}$$

In order to obtain an expression for $[\nabla/\nabla x(s)]^n\ [\tau_n(s)\rho_n(s)]$ we successively apply the formula $\nabla f(s)g(s) = f(s)\nabla g(s) + g(s-1)\nabla f(s)$, as well as formulae

$$\frac{\Delta \tau_n(s)}{\Delta x(s)} = q^n \tau_n' \qquad \left[\frac{\nabla}{\nabla s(s-1)}\right]^n = q^{2n} \left[\frac{\nabla}{\nabla x(s)}\right]^n.$$

Then, equation (11) gives us the following:

$$P_{n+1}(s)_q = \frac{q^{-n(n+1)/2} B_{n+1}}{\rho(s)}$$
$$\times \left(\tau_n(s) \left[\frac{\nabla}{\nabla x(s)}\right]^n [\rho_n(s)] + q^{2n-1}[n]_q \tau_n' \left[\frac{\nabla}{\nabla x(s)}\right]^{n-1} [\rho_n(s-1)] \right). \tag{12}$$

Using the Rodriques formula for the difference derivative of the polynomial ([13], p 66, equation (3.2.18)) we find (notice that $\Delta x(s-1) = q^{-2}\Delta x(s)$)

$$\frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\Delta P_n(s-1)_q}{\Delta x(s-1)} = \frac{-q^{-(n-1)(n+2)/2}\lambda_n B_n}{\sigma(s)\rho(s)} \left[\frac{\nabla}{\nabla x(s-1)}\right]^{n-1} [\rho_n(s-1)]$$
$$= \frac{-q^{-(n-1)(n-2)/2}\lambda_n B_n}{\sigma(s)\rho(s)} \left[\frac{\nabla}{\nabla x(s)}\right]^{n-1} [\rho_n(s-1)].$$

Therefore, equation (12) can be rewritten in the form

$$P_{n+1}(s)_q = \frac{B_{n+1}\tau_n(s)}{B_n} P_n(s)_q - \frac{[n]_q B_{n+1}\tau_n'\sigma(s)}{\lambda_n B_n} \frac{\nabla P_n(s)_q}{\nabla x(s)}$$

and then, the following differentiation formula holds:

$$\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q \tau_n'} \left[\tau_n(s) P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q\right]. \tag{13}$$

If we now use the power expansion of $\tau_n(s)$, i.e., $\tau_n(s) = \tau_n' x_n(s) + \tau_n(0) = \tau_n' q^n x(s) + \tau_n(0)$ and the TTRR (7) we obtain the *first structure relation*

$$\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} = \tilde{S}_n P_{n+1}(s)_q + \tilde{T}_n P_n(s)_q + \tilde{R}_n P_{n-1}(s)_q \tag{14}$$

where

$$\tilde{S}_n = \frac{\lambda_n}{[n]_q}\left[q^n \alpha_n - \frac{B_n}{\tau_n' B_{n+1}}\right] \qquad \tilde{T}_n = \frac{\lambda_n}{[n]_q}\left[q^n \beta_n - \frac{\tau_n(0)}{\tau_n'}\right] \tag{15}$$
$$\tilde{R}_n = \frac{\lambda_n q^n \gamma_n}{[n]_q}.$$

2.2. The second structure relation for the q-polynomials in the lattice $x(s) = q^{2s}$

Let us try to obtain now the second structure relation. First, we notice that

$$\Delta \frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\Delta P_n(s)_q}{\Delta x(s)} - \frac{\nabla P_n(s)_q}{\nabla x(s)}.$$

Then, by using the difference equation (2)

$$\begin{aligned}\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} &= \sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} - \sigma(s)\Delta\frac{\nabla P_n(s)_q}{\nabla x(s)} \\ &= [\sigma(s) + \tau(s)\Delta x(s-\tfrac{1}{2})]\frac{\nabla P_n(s)_q}{\nabla x(s)} + \lambda_n \Delta x(s-\tfrac{1}{2}) P_n(s)_q\end{aligned}$$

and (14) we find

$$\begin{aligned}[\sigma(s) + \tau(s)\Delta x(s-\tfrac{1}{2})]\frac{\Delta P_n(s)_q}{\Delta x(s)} &= \tilde{S}_n P_{n+1}(s)_q + (\tilde{T}_n - \lambda_n \Delta x(s-\tfrac{1}{2})) P_n(s)_q \\ &\quad + \tilde{R}_n P_{n-1}(s)_q.\end{aligned} \tag{16}$$

Now, taking into account the fact that $\Delta x(s-\frac{1}{2}) = (q-q^{-1})x(s)$, and using the TTRR (7) we finally obtain the *second structure relation*

$$[\sigma(s) + \tau(s)\Delta x(s-\tfrac{1}{2})]\frac{\nabla P_n(s)_q}{\nabla x(s)} = S_n P_{n+1}(s)_q + T_n P_n(s)_q + R_n P_{n-1}(s)_q \tag{17}$$

where

$$S_n = \tilde{S}_n - (q-q^{-1})\lambda_n \alpha_n \qquad T_n = \tilde{T}_n - (q-q^{-1})\lambda_n \beta_n \qquad R_n = \tilde{R}_n - (q-q^{-1})\lambda_n \gamma_n. \tag{18}$$

3. Recurrence relations for connection coefficients

Let us consider two families of q-polynomials $P_n(x)$ and $Q_n(x)$ belonging to the class of discrete orthogonal polynomials in the exponential lattice $x(s) = q^{2s}$. Each polynomial $P_n(x)$ can be represented as a linear combination of the polynomials $Q_n(x)$. In particular,

$$P_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x). \tag{19}$$

For the family $P_n(x)$ we will use the notation

(i) $\sigma(s)$, $\tau(s)$ and λ_n for the difference equation (2),

(ii) α_n, β_n and γ_n for the TTRR (7) coefficients,

(iii) S_n, R_n and T_n for the second structure relation (17),

and for the $Q_n(x)$

(i) $\bar\sigma(s)$, $\bar\tau(s)$ and $\bar\lambda_n$ for the difference equation (2),

(ii) $\bar\alpha_n$, $\bar\beta_n$ and $\bar\gamma_n$ for the TTRR (7) coefficients,

(iii) $\bar S_n$, $\bar R_n$ and $\bar T_n$ for the second structure relation (17).

Since the polynomials of the family $P_n(x)$ are solutions of the second-order difference equation (2) the action of the difference operator of second order $\hat L$, defined by

$$\hat L = \sigma(s)\frac{\Delta}{\Delta x(s-\frac12)}\left[\frac{\nabla}{\nabla x(s)}\right] + \tau(s)\frac{\Delta}{\Delta x(s)} + \lambda_n$$

in equation (19) gives

$$\sum_{m=0}^{n} C_m(n)\left[\sigma(s)\frac{\Delta}{\Delta x(s-\frac12)}\left[\frac{\nabla Q_m(s)}{\nabla x(s)}\right] + \tau(s)\frac{\Delta Q_m(x)}{\Delta x(s)} + \lambda_n Q_m(x)\right] = 0. \tag{20}$$

Multiplying by $\bar\sigma(s)$ and using

$$\bar\sigma(s)\frac{\Delta}{\Delta x(s-\frac12)}\left[\frac{\nabla Q_m(x)}{\nabla x(s)}\right] = -\bar\tau(s)\frac{\Delta Q_m(x)}{\Delta x(s)} - \bar\lambda_n Q_m(x)$$

we obtain the relation

$$\sum_{m=0}^{n} C_m(n)\left[(\tau(s)\bar\sigma(s) - \bar\tau(s)\sigma(s))\frac{\Delta Q_m(x)}{\Delta x(s)} + (\lambda_n\bar\sigma(s) - \sigma(s)\bar\lambda_m)Q_m(x)\right] = 0. \tag{21}$$

In order to eliminate $\Delta Q_m(x)/\Delta x(s)$, we multiply (21) by $\bar\sigma(s) + \bar\tau(s)\Delta x(s-\frac12)$ and use the second structure relation (17) for the $Q_m(x)$ family, obtaining

$$\sum_{m=0}^{n} C_m(n)[(\tau(s)\bar\sigma(s) - \bar\tau(s)\sigma(s))(\bar S_m Q_{m+1}(x) + \bar R_m Q_{m-1}(x) + \bar T_m Q_m(x)) \\ +(\bar\sigma(s) + \bar\tau(s)\Delta x(s-\tfrac12))(\lambda_n\bar\sigma(s) - \sigma(s)\bar\lambda_m)Q_m(x)] = 0. \tag{22}$$

The last step consists of expanding the remaining terms of type $\bar\sigma^2(s)Q_m(x)$, $\bar\sigma(s)\sigma(s)Q_m(x)$, $\sigma(s)\bar\tau(s)Q_m(x)$ and $\bar\sigma(s)\tau(s)Q_m(x)$ in a linear combination of $Q_n(x)$ by using the TTRR (7) repeatedly for the $Q_n(x)$ family.

After this process, (2) reduces to

$$\sum_{m=0}^{N} M_m[C_0(n), C_1(n), \ldots, C_n(n)]Q_m(x) \tag{23}$$

where

$$N = \max\{n + \deg\sigma + \deg(\bar\sigma), n + 2\deg(\bar\sigma), n + 1 + \deg(\bar\sigma) + \deg(\tau), n + 1 + \deg(\bar\tau) \\ + \deg(\sigma), 1 + \deg(\bar\tau) + \deg(\bar\sigma)\}.$$

Taking into account the linear independent of the family $Q_m(x)$ we obtain the linear system

$$M_m[C_0(n), C_1(n), \ldots, C_n(n)] = 0. \tag{24}$$

These relations contain (linearly) several connection coefficients $C_i(n)$ depending essentially on the degrees of $\sigma(s)$ and $\bar{\sigma}(s)$. In the most general situation they are polynomials of second degree in $x(s) = q^{2s}$. In this case, we obtain a relation of the type of linear system that we are looking,

$$M_m[C_{m+4}(n), \ldots, C_{m-4}(n)] = 0 \tag{25}$$

which is valid for n greater than or equal to the number of initial conditions needed to start the recursion ($n \geqslant 8$). Notice that for ($n < 8$) the system also gives the solution, but not in a recurrent way.

Notice that for the q-Hahn, q-Meixner, q-Charlier and q-Kravchuk polynomials, as is shown in [13], p 95, table 3.3, $\sigma(s)$ is a polynomial of second degree in $x(s) = q^{2s}$. This implies that for such polynomials the recurrence relations for the connection coefficient are all of the form (25). Again note that we follow the notation introduced by Nikiforov *et al* [13].

4. Recurrence relations for connection coefficients: a simple example

As we have noted in the previous section, the recurrence relation for the connection coefficients for different classes of q-polynomials are too large (eight terms). Here we will analyse a more simple case. First, notice that in the previous algorithm we have not used the orthogonality property of the polynomials P_n, but only that they satisfy a difference equation. On the other hand, for the polynomials Q_m we need to have structure relations as well as three-term recurrence relations. Let us show an example in which we decompose a set of polynomials $P_n(s)$, satisfying a certain difference equation of first order in the lattice $x(s) = q^{2s}$, as a linear combination of orthogonal q-polynomials defined in the same lattice, i.e. the q-Hahn, q-Meixner, q-Kravchuk and q-Charlier orthogonal polynomials (see [13, 4, 17]).

Let us define the quantities $(s)_q$ and $(s_n)_q$ by

$$(s)_q = \frac{q^{2s} - 1}{q^2 - 1} = q^{s-1}[s]_q \tag{26}$$

and

$$(s_n)_q = (s)_q(s-1)_q \cdots (s-n+1)_q - \prod_{k=0}^{n-1} \frac{q^{2s+2k} - 1}{q^2 - 1}. \tag{27}$$

The quantities $(s_n)_q$ are closely related to the *q-Stirling numbers* $\tilde{S}_{q^2}(n,k)$, $s^*_{q^2}(n,k)$ [21] by formulae

$$(s)_q^n = \sum_{k=0}^{n} \tilde{S}_{q^2}(n,k)(s_k)_q \qquad (s_n)_q = \sum_{k=0}^{n} s^*_{q^2}(n,k)(s)_q^k \tag{28}$$

and satisfy the following two difference equations (here, as before, $x(s) = q^{2s}$):

$$(q^{2s} - 1)\frac{\nabla(s_n)_q}{\nabla x(s)} - q^{-n+1}[n]_q(s_n)_q = 0 \tag{29}$$

and

$$(q^{2s-2n+2}-1)\frac{\nabla(s_n)_q}{\nabla x(s)}-q^{-n+1}[n]_q(s_n)_q=0. \tag{30}$$

Since $(s_n)_q$ is a polynomial in $x(s)=q^{2s}$, it can be represented as a linear combination of the polynomials $Q_m(x)$, the q-polynomials in the exponential lattice. In particular,

$$(s_n)_q=\sum_{m=0}^{n}C_m(n)Q_m(x). \tag{31}$$

Let us obtain the recurrence relation for the connection coefficients $C_m(n)$ between the $(s_n)_q$, and the q-Charlier, q-Meixner or q-Kravchuk cases. (For q-Hahn polynomials we will consider a separate relation.) In order to do this we apply the operator

$$\bar{\mathcal{L}}=(q^{2s}-1)\frac{\nabla}{\nabla x(s)}-q^{-n+1}[n]_q \tag{32}$$

to both sides of (31). Using formula (29) ($\tilde{\mathcal{L}}(s_n)_q=0$) and multiplying by q^{2s} we obtain the following expression:

$$0=\sum_{m=0}^{n}C_m(n)\left\{q^{2s}(q^{2s}-1)\frac{\nabla Q_m(x)}{\nabla x(s)}-q^{-m+1}[m]_q q^{2s}Q_m(x)\right\}. \tag{33}$$

Taking into account the fact that for q-Charlier, q-Meixner and q-Kravchuk polynomials the $\sigma(s)$ function in (2) coincides with $q^{2s}(q^{2s}-1)$ and applying the structure relation (14) and the TTRR (7) to the previous expression we find

$$0=\sum_{m=0}^{n}C_m(n)\{A_mQ_{m+1}(x)+B_mQ_m(x)+\Gamma_mQ_{m-1}(x)\}$$

from which we obtain the following TTRR for the connection coefficients $C_m(n)$:

$$A_{m-1}C_{m-1}(n)+B_mQ_m(n)+\Gamma_{m+1}C_{m+1}(n)=0 \tag{34}$$

where

$$\begin{aligned}
A_{m-1}&=\tilde{S}_{m-1}-q^{-m+2}[m-1]_q\alpha_{m-1}=\frac{\lambda_{m-1}}{[m-1]_q}\left[q^{m-1}\alpha_{m-1}-\frac{B_{m-1}}{\tau'_{m-1}B_m}\right]\\
&\quad -q^{-m+2}[m-1]_q\alpha_{m-1}\\
B_m&=\tilde{T}_m-q^{-m+1}[m]_q\beta_m=\frac{\lambda_m}{[m]_q}\left[q^m\beta_m-\frac{\tau_m(0)}{\tau'_m}\right]-q^{-m+1}[m]_q\beta_m\\
\Gamma_{m+1}&=\tilde{R}_{m+1}-q^{-m}\gamma_{m+1}=\frac{\lambda_{m+1}q^{m+1}\gamma_{m+1}}{[m-1]_q}-q^{-m}[m+1]_q\gamma_{m+1}.
\end{aligned} \tag{35}$$

In order to obtain the recurrence relation for the connection coefficients in the q-Hahn case we apply the operator $\tilde{\mathcal{L}}(32)$ to both sides of (31). Tasking into account the fact that $\tilde{\mathcal{L}}(s_n)_q=0$ and multiplying by $q^{2\alpha+2N}-q^{2s}$ we obtain the following expression:

$$0=\sum_{m=0}^{n}C_m(n)\left\{(q^{2\alpha+2N}-q^{2s})(q^{2s}-1)\frac{\nabla Q_m(x)}{\nabla x(s)}-q^{-m+1}[m]_q q^{2s}Q_m(s)\right\}. \tag{36}$$

Taking into account the fact that for the q-Hahn case the $\sigma(s)$ function in (2) coincides with $(q^{2\alpha+2N}-q^{2s})(q^{2s}-1)$ (see [17]) and using the structure relation (14) and the TTRR (7) we obtain the same expression (34) as before for the TTRR for the connection coefficients $C_m(n)$, where now

$$
\begin{aligned}
A_{m-1} &= \tilde{S}_{m-1} + q^{-m+2}[m-1]_q \alpha_{m-1} = \frac{\lambda_{m-1}}{[m-1]_q}\left[q^{m-1}\alpha_{m-1} - \frac{B_{m-1}}{\tau'_{m-1}B_m}\right] \\
&\quad -q^{-m+2}[m-1]_q\alpha_{m-1} \\
B_m &= \tilde{T}_m + q^{-m+1}[m]_q\beta_m - [m]_q q^{2N+2\alpha-m+1} = \frac{\lambda_m}{[m]_q}\left[q^m\beta_m - \frac{\tau_m(0)}{\tau'_m}\right] \\
&\quad +q^{-m+1}[m]_q\beta_m - [m]_q q^{2N+2\alpha-m+1} \\
\Gamma_{m+1} &= \tilde{R}_{m+1} + q^{-m}\gamma_{m+1} = \frac{\lambda_{m+1}q^{m+1}\gamma_{m+1}}{[m+1]_q} + q^{-m}[m+1]_q\gamma_{m+1}.
\end{aligned}
\tag{37}
$$

4.1. The three-term recurrence relation for connection coefficients of the q-powers $(s_n)_q$ and the q-Meixner polynomials $m_n^{\gamma,\mu}(s,q)$

Here we will calculate the coefficients A_{m-1}, B_m and Γ_{m+1} of the three-term recurrence relation for connection coefficients $C_m(n)$ (34) of the q-powers $(s_n)_q$ and the q-Meixner polynomials $m_n^{\gamma,\mu}(s,q)$, i.e.

$$(s_n)_q = \sum_{k=0}^{n} C_m(n) m_k^{\gamma,\mu}(s,q).$$

The main data for the q-Meixner polynomials are provided in [4]. In our work we will use monic polynomials, i.e. the leading coefficient $a_n = 1$. In table 1 we provide the quantities needed for our calculations. (For more details see [4, 13]). We want to point out that these monic q-Meixner polynomials $m_n^{\gamma,\mu}(s,q)$ [4] are connected with the monic little q-Jacobi polynomials $p_n(x; a, b|q)$ [6, 8] by the relation

$$m_n^{\gamma,\mu}(s,q) = p_n(q^{2s}; \mu, q^{2\gamma-2}|q^2).$$

Table 1. The main data for q-Meixner polynomials [4].

	$m_n^{\gamma,\mu}(s,q)$, $\mu = q^{2\theta}$
$\sigma(s)$	$q^{2s}(q^{2s}-1)$
$\tau(s)$	$q^{s+2\theta+\gamma+2}[s+\gamma]_q - q^2[s]_q$
λ_n	$-[n]_q q^{\gamma+\theta+1}[n+\gamma+\theta]_q$
τ'_n	$q^{\gamma+\theta+1}[2n+\gamma+\theta+1]_q$
$\tau_n(0)$	$-q^{\theta+1}[n+\theta+1]_q$
$\frac{B_n}{B_{n+1}}$	$-q^{\gamma+\theta+1}\frac{[2n+\gamma+\theta+1]_q[2n+\gamma+\theta]_q}{[n+\gamma+\theta]_q}$
α_n	1
β_n	$q^{-\gamma}\frac{[n+1]_q[n+\theta+1]_q}{[2n+\gamma+\theta+1]_q} - q^{-\gamma}\frac{[n]_q[n+\theta]_q}{[2n+\gamma+\theta-1]_q}$
γ_n	$\frac{q^{-n-3\gamma+2}[n]_q[\gamma+n-1]_q[n+\gamma+\theta-1]_q[n+\theta]_q}{[2n+\gamma+\theta-2]_q[2n+\gamma+\theta-1]_q^2[2n+\gamma+\theta]_q}$

If we now apply formulae (15) and (14) we obtain for q-Meixner polynomials the structure relation

$$\sigma(s)\frac{\nabla m_n^{\gamma,\mu}(s,q)}{\nabla x(s)} = \tilde{S}_n m_{n+1}^{\gamma,\mu}(s,q) + \tilde{T}_n m_n^{\gamma,\mu}(s,q) + \tilde{R}_n m_{n-1}^{\gamma,\mu}(s,q) \tag{38}$$

where

$$\tilde{S}_n = -q^{\gamma+\theta+1}(q^n[n+\gamma+\theta]_q + [2n+\gamma+\theta]_q)$$
$$\tilde{T}_n = -q^{n+\theta+1}[n+\gamma+\theta]_q\left(\frac{n+1]_q[n+\theta+1]_q}{[2n+\gamma+\theta+1]_q} - \frac{[n]_q[n+\theta]_q}{[2n+\gamma+\theta-1]_q}\right)$$
$$-\frac{q^{\theta+1}[n+\theta+1]_q[n+\gamma+\theta]_q}{[2n+\gamma+\theta+1]_q}$$
$$\tilde{R}_n = -\frac{q^{-2\gamma+\theta+3}[n]_q[\gamma+n-1]_q[n+\gamma+\theta]_q[n+\gamma+\theta-1]_q[n+\theta]_q}{[2n+\gamma+\theta-2]_q[2n+\gamma+\theta-1]_q^2[2n+\gamma+\theta]_q}. \tag{39}$$

Then, by using (35) we finally find the coefficients A_{m-1}, B_m and Γ_{m+1}:

$$A_{m-1} = -q^{\gamma+\theta+1}(q^{m-1}[m+\gamma+\theta-1]_q + [2m+\gamma+\theta-2]_q) - q^{-m+2}[m-1]_q \tag{40}$$
$$B_m = -\left(\frac{[m+1]_q[m+\theta+1]_q}{[2m+\gamma+\theta+1]_q} - \frac{[m]_q[m+\theta]_q}{[2m+\gamma+\theta-1]_q}\right)q^{\theta+1}[2m+\gamma+\theta]_q$$
$$-\frac{q^{\theta+1}[m+\gamma+\theta]_q[m+\theta+1]_q}{[2m+\gamma+\theta+1]_q} \tag{41}$$
$$\Gamma_{m+1} = -\frac{q^{-m-2\gamma+\theta+2}[m+1]_q[\gamma+m]_q[m+\gamma+\theta]_q[m+\theta+1]_q}{[2m+\gamma+\theta]_q[2m+\gamma+\theta+1]_q[2m+\gamma+\theta+2]_q}. \tag{42}$$

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